Lecture 22

Wednesday, November 27, 2019 10:15 AM

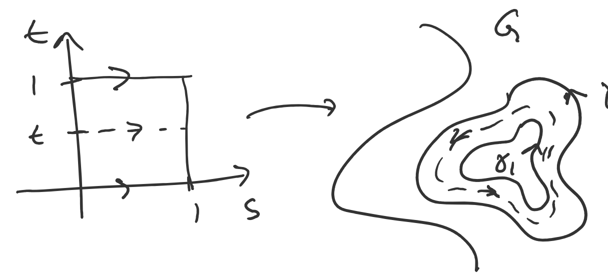
Recall. • Cauchy's IF–II: Suppose $\gamma_1, \dots, \gamma_n$ closed curves in G s.t. $n(\gamma_1, z) + \dots + n(\gamma_n, z) = 0$, $\forall z \in \mathbb{C} \setminus G$. Then, if f anal. in G,

$$f(z) \sum_{k=1}^{n} n(\gamma_k, z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z} d\zeta .$$

• The condition $(*)$ $\sum_{k=1}^{n} n(\gamma_k, z) = 0$, $\forall z \in \mathbb{C} \setminus G$ is related to a condition in algebraic topology. The formal sum $\sigma = \sum_{k=1}^{n} \gamma_k$ is a "1-chain", and the condition $(*)$ means σ is <u>homologous to 0</u> in G, $\sigma \approx 0$ in G. If $n = 1$, we say the curve γ_1 is homo. to 0. In a.t., $\sigma \approx 0$ in G means it is the "boundary" of a "2-chain" in G. In this course, $\sigma \approx 0$ in G means $(*)$.

Homotopies and a homotopic version of Cauchy's Thm.

Def. Let $\gamma_0, \gamma_2 : [0,1] \to G \subseteq \mathbb{C}$ be closed curves. γ_0 is <u>homotopic in G</u> to γ_1 if $\exists$ cont. map $\Gamma : [0,1] \times [0,1] \to G$ s.t. $\Gamma(\cdot, 0) = \gamma_0$, $\Gamma(\cdot, 1) = \gamma_1$ and $\Gamma(0,t) = \Gamma(1,t)$, $\forall t \in [0,1]$.



Rem. • Often use $\Gamma_t := \Gamma(\cdot, t)$. Note Γ_t is a curve, but we <u>do not</u> require any smoothness, even if γ_0, γ_1 are p-w smooth.

• Notation for homotopies, $\gamma_0 \sim \gamma_1$ in G or if G is understood, $\gamma_0 \sim \gamma_1$.

Basic fact. (Ex). $\sim$ is an equivalence relation. If $\gamma_0 \sim \gamma_1$, $\gamma_1 \sim \gamma_2$, then $\gamma_1 \sim \gamma_0$ and $\gamma_0 \sim \gamma_2$.

Cauchy's Thm–III. Let f be anal. in $G \subseteq \mathbb{C}$, and $\gamma_0, \gamma_1 : [0,1] \to G$ p-w smooth curves s.t. $\gamma_0 \sim \gamma_1$ in G. Then, $\int_{\gamma_0} f dz = \int_{\gamma_1} f dz$.

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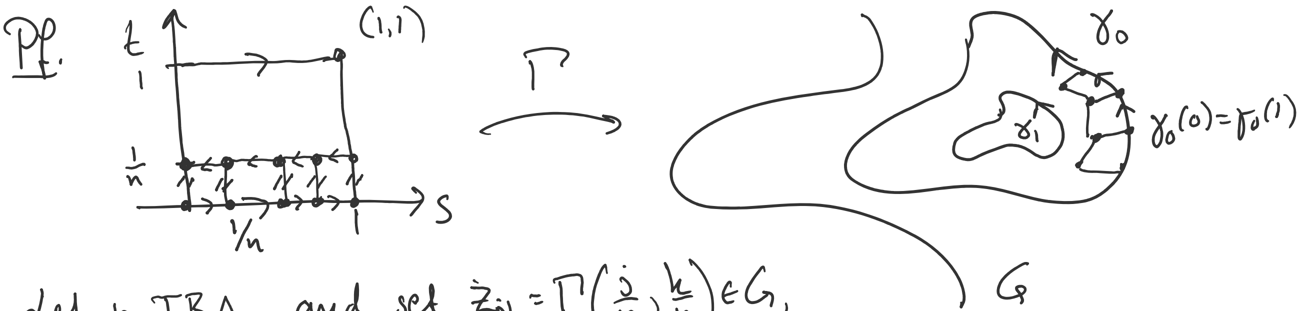
Pf.



let n TBA, and set $z_{jk} = \Gamma\left(\frac{j}{n}, \frac{k}{n}\right) \in G$,
and P_{jk} the polygonal path $\Big[\underbrace{\Gamma\left(\tfrac{j}{n},\tfrac{k}{n}\right)}_{z_{jk}}, \underbrace{\Gamma\left(\tfrac{j+1}{n},\tfrac{k}{n}\right)}_{z_{j+1,k}}\Big] \cup \Big[\underbrace{\Gamma\left(\tfrac{j+1}{n},\tfrac{k}{n}\right)}_{z_{j+1,k}}, \underbrace{\Gamma\left(\tfrac{j+1}{n},\tfrac{k+1}{n}\right)}_{z_{j+1,k+1}}\Big] \cup$
$\Big[\underbrace{\Gamma\left(\tfrac{j+1}{n},\tfrac{k+1}{n}\right)}_{z_{j+1,k+1}}, \underbrace{\Gamma\left(\tfrac{j}{n},\tfrac{k+1}{n}\right)}_{z_{j,k+1}}\Big] \cup \Big[\underbrace{\Gamma\left(\tfrac{j}{n},\tfrac{k+1}{n}\right)}_{z_{j,k+1}}, \underbrace{\Gamma\left(\tfrac{j}{n},\tfrac{k}{n}\right)}_{z_{j,k}}\Big]$, and P_k the polygonal path $\left[\Gamma\left(0,\tfrac{k}{n}\right), \Gamma\left(\tfrac{1}{n},\tfrac{k}{n}\right)\right] \cup \dots \cup \left[\Gamma\left(\tfrac{n-1}{n},\tfrac{k}{n}\right), \Gamma\left(1,\tfrac{k}{n}\right)\right]$

Observe (look at picture) that if $Q_{jk} := \Gamma\left(\left[\frac{j}{n},\frac{j+1}{n}\right] \times \left[\frac{k}{n},\frac{k+1}{n}\right]\right)$ is contained in $B(z_{jk},\varepsilon) \subseteq G$, then $\displaystyle\int_{P_{jk}} f\,dz = 0$, $\forall\, j,k \in \{0,\dots,n\}^n$.

Baby Cauchy

Summing over j and noting that integrals over "up/down" segments cancel $\Rightarrow$ $\displaystyle\int_{P_k} f\,dz = \int_{P_{k+1}} f\,dz$, for $k = 0,\dots,n-1$.

Next, note that if we let $\mu_j = \gamma_0\big|_{\left[\frac{j}{n},\frac{j+1}{n}\right]}$ for $j = 0,\dots,n-1$, then under the assumption on Q_{jk} above ($\Rightarrow Q_{j,0} \subseteq B(z_{j,0},\varepsilon) \subseteq G$)

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$\displaystyle\int_{\sigma_j} f\,dz = \int_{[z_{j,0},\, z_{j+1,0}]} f\,dz$. Again, summing $\Rightarrow \displaystyle\int_{\gamma_0} f\,dz = \int_{P_0} f\,dz$.

Similarly, $\displaystyle\int_{\gamma_1} f\,dz = \int_{P_{n-1}} f\,dz$. We conclude $\displaystyle\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$.

Thus, remains to show $\exists\, \varepsilon > 0$ and n s.t. $Q_{jk} \subseteq B(z_{jk},\varepsilon) \subseteq G$ for $j,k \in \{0,1,\dots,n-1\}^2$.

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$I^2 = [0,1]\times[0,1]$ is compact, Γ cont. $\Rightarrow$ Γ unif. cont. + $\Gamma(I^2)$ is compact. $\Rightarrow$ $\exists\, \varepsilon>0$ s.t. $d(\Gamma(I^2), \partial G) = \varepsilon \Rightarrow B(z_{j,k}, \varepsilon) \subseteq G$.

Unif. cont. $\Rightarrow$ $\exists\, \delta>0$ s.t. $|\Gamma(s,t)-\Gamma(s',t')| < \varepsilon$ when $|(s,t)-(s',t')|<\delta$. Since $\operatorname{diam}\Big(\underbrace{[\tfrac{j}{n},\tfrac{j+1}{n}]\times[\tfrac{k}{n},\tfrac{k+1}{n}]}\Big) = \sqrt{2}/n$,

we can choose n s.t. $|(s,t)-(s',t')|<\delta$ when $(s,t),(s',t) \in$ ⌄.

This completes the proof.

<u>Cor.</u> (Cauchy's Thm – II). Let $\gamma:[0,1]\to G \subseteq \mathbb{C}$ be homotopic in G to a constant curve ($\gamma\sim 0$ in G). If f is anal. in G, then

$$\int_\gamma f\,dz = 0.$$

<u>Rem.</u> ◦ CT-II is actually weaker than CT-I, since if $\gamma\sim 0$ in G $\Rightarrow$

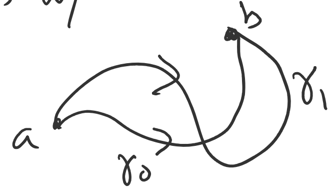
$\forall a \in \mathbb{C}\setminus G$, $f(z) = \frac{1}{z-a}$ is anal. in G $\Rightarrow$

$$n(\gamma,a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = 0 \text{ by CT-II.} \Rightarrow \gamma \approx 0 \text{ in } G$$

(Converse is not true.) However, the homotopy condition is more "geometric".

<u>Fixed End Point (FEP) homotopies.</u>

<u>Def.</u> If $\gamma_0,\gamma_1:[0,1]\to G$ are curves w/ same endpts $\gamma_0(0)=\gamma_1(0)=a$, $\gamma_0(1)=\gamma_1(1)=b$



Then γ_0 is FEP-homotopic to γ_1 (in G) if $\exists$ cont. map $\Gamma:[0,1]^2\to G$ s.t. $\Gamma(\cdot,0)=\gamma_0$, $\Gamma(\cdot,1)=\gamma_1$, and $\Gamma(0,t)=a$, $\Gamma(1,t)=b$, $\forall t\in[0,1]$.

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s.t. $\Gamma(\cdot,0) = \gamma_0$, $\Gamma(\cdot,1)=\gamma_1$, and $\Gamma(0,t)=a$, $\Gamma(1,t)=b$, $\forall t\in[0,1]$.

<u>Notation</u>: $\gamma_0 \underset{FEP}{\sim} \gamma_1$ in G.

, $\Gamma_t(s) = \Gamma(s,t)$ curve (no smoothness) from a to b.

<u>Lemma 1</u>. If $\gamma_0 \underset{FEP}{\sim} \gamma_1$ in G, then $\gamma := \gamma_0 - \gamma_1$ is ~ 0 in G.

<u>Pf by Pic</u>.

Define $\tilde{\Gamma}(\tilde{s},\tilde{t})$ homotopy of $\gamma = \gamma_0 - \gamma_1$ to 0 as in pic.

Pf.

<u>Thm 1 (Independence of path)</u>. Let $\gamma_0, \gamma_1 : [0,1] \to G \subseteq \mathbb{C}$ be p-w smooth curves from a to b. If $\gamma_0 \underset{FEP}{\sim} \gamma_1$ in G, then for any f anal. in $G \Rightarrow$

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz .$$

<u>Pf</u>. $\gamma = \gamma_0 - \gamma_1 \sim 0$ in $G \underset{CT\text{-}II}{\Rightarrow} \int_\gamma f\,dz = 0 \Rightarrow$ $\square$